C^*-Algebra Valued Modular G-Metric Spaces with Applications in Fixed Point Theory

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Abstract: This article introduces a new type of C^*-algebra valued modular G-metric spaces that is more general than both C^*-algebra valued modular metric spaces and modular G-metric spaces. Some properties are also discussed with examples. A few common fixed point results in C^*-algebra valued modular G-metric spaces are discussed using the "C_*-class function", along with some suitable examples to validate the results. Ulam–Hyers stability is used to check the stability of some fixed point results. As applications, the existence and uniqueness of solutions for a particular problem in dynamical programming and a system of nonlinear integral equations are provided.

1. Introduction

In recent years, C^*-algebra has attracted a lot of interest due to its prospective applications in modern mathematics, entropy analysis, fixed point theory, noncommutative geometry, string theory, quantum mechanics, and other fields. Let $\mathbb{A}$ be a Banach algebra and '$*$' be involution self-mapping on $\mathbb{A}$. Then $\mathbb{A}$ is said to be a C^*-algebra if it satisfies for any $z_1, z_2 \in \mathbb{A}$ and $\alpha, \mu \in \mathbb{C}$: ([1,2]) $z_1^{**} = z_1$, $(z_1 z_2)^* = z_2^* z_1^*$, $(\alpha z_1 + \mu z_2)^* = \bar{\alpha} z_1^* + \bar{\mu} z_2^*$ and $\|z_1^* z_1\| = \|z_1\|^2$, (which easily shows that $\|z_1^*\| = \|z_1\|$). Let $\mathbb{A}$ be an unital C^*-algebra with the identity $1_{\mathbb{A}}$ and zero element θ. Every element of the set $\mathbb{A}_+ = \{x \in \mathbb{A} : x \succeq \theta\}$ is called positive, any element of A_+ is $x \in A$ with $x = x^*$ and spectrum $\sigma(x) \subset \mathbb{R}_+$, where $\sigma(x) = \{\alpha \in \mathbb{R} : |\alpha 1_{\mathbb{A}} - x| = 0\}$.

A partial ordering "$\succeq$" on $\mathbb{A}$ behaves as $x \succeq y \Leftrightarrow x - y \in \mathbb{A}_+$. For every element $\theta \preceq x \in \mathbb{A}$ has a unique positive square root, i.e., $|x| = (x^* x)^{\frac{1}{2}}$.

Ma et al. [3] initiated C^*-*avMS* by replacing real numbers with positive elements of unital C^*-algebra, which generalizes metric spaces, and studied some fixed point results. Ma et al. [4] also generalized this concept and pioneered C^*-*avb-MS*. According to Alsulami et al. [5] and Kadelburg et al. [6], the fixed point results in C^*-*avMS* and C^*-*avb-MS* can be found as the implications of their classic metric spaces and *b*-metric spaces, respectively. Despite this, Mustafa et al. [7] described the significance and obstacles of studying fixed

point theory in C^*-algebra, and how research on such spaces has become more popular among researchers. Recently, Kumar et al. [8] explained some fixed point results via "C_*-class function" in C^*-*avMS*, which are more general than metric spaces. Due to the importance of the study of fixed point results in the setting of C^*-algebra, researchers have initiated more new generalized spaces than metric spaces, such as C^*-*avGMS* [9], C^*-*avG_bMS* [10], C^*-*avSMS* [11], C^*-algebra valued partial metric spaces [12], etc., which enriches this field (see also [13–28]).

Chistyakov [29,30] initiated modular metric spaces. Since then researchers developed fixed point theory in these spaces. For instance, Zhu et al. [31] studied some fixed point results on asymptotic pointwise contractions in modular metric spaces that generalize metric spaces; Okeke et al. [32] introduced a few fixed point results for rational contractive mappings in modular metric spaces with applications in integrodifferential equations; Shateri [33] initiated C^*-algebra valued modular spaces that generalize modular spaces. Ege et al. [34] initiated modular *b*-metric spaces and applied these concepts in fixed point theory. Based on these papers, Moeini et al. [2,35,36] initiated C^*-algebra valued modular metric spaces and Das and Mishra [37] introduced C^*-algebra valued modular *b*-metric spaces.

Hyers [38] answered Ulam's [39] question about the stability of functional equations for Banach spaces, and the stability used for the answer is known as Ulam–Hyers stability. Studies addressing Ulam–Hyers stability results and different stability results in fixed point theory can be seen in [40–50].

Mustafa and Sims [51] initiated *G*-metric spaces as a metric space generalization and studied some fixed point results. Researchers developed the study densely for *G*-metric spaces in fixed point theory, some of which can be seen in [52–58].

Jleli et al. [59] and Samet et al. [60] showed that fixed point results in *G*-metric spaces can be generated from existence results in the context of quasi metric spaces.

Asadi et al. [61,62] proved some results in *G*-metric spaces which cannot be obtained from the existence result in the environment of metric space. Agarwal et al. [63] showed that if the contractivity condition of the fixed point result on a *G*-metric space can be simplified to two variables then an analogous fixed point result in the context of classic metric spaces can be established easily. They also constructed some new fixed point results that cannot be reduced to quasi metric spaces, with new contractive conditions.

Sedghi et al. [64] introduced *S*-metric spaces and claimed that space is a generalization of *G*-metric spaces, but Dung et al. [65] explained that this was incorrect. As a result, studying in such environments is both exciting and demanding.

Due to the demand for research in modular metric spaces, Azadifar et al. [66] initiated modular *G*-metric spaces, which generalize modular metric spaces as well as metric spaces. Azadifar et al. [67] also studied common fixed point results in modular *G*-metric spaces. Okeke et al. [68] studied some fixed point results in modular *G*-metric spaces and Okeke et al. [69] studied in preordered modular *G*-metric spaces, which were applied to solve nonlinear integral equations.

From the above study, it can be observed that, along with different generalized metric spaces, *G*-metric spaces and modular *G*-metric spaces have numerous applications in fixed point theory. Furthermore, studying the fixed point theorem in the context of C^*-algebra has a wide range of applications. Researchers present many applications from the obtained results by expressing multiple situations that exemplify the application domains in distinct generalized C^*-*avMS* as well as generalized modular metric spaces. The foregoing research leads us to investigate modular *G*-metric spaces in C^*-algebra in order to generalize the existing spaces. The study of such spaces resulted in the generalization of modular *G*-metric spaces as well as C^*-*avGMS*. Hence, all the results in C^*-*avGMS* are automatically generalized compared to the existing spaces mentioned in the above literature.

In this paper, we introduce C^*-algebra valued modular *G*-metric spaces via "C_*-class function" to generalize the fixed point results and to offer possible improvements on the structures of some types of metric spaces in algebraic topology. Some fixed point results

are discussed with suitable examples, and the stability of these results is checked by using Ulam–Hyers stability. Applications for existence and uniqueness results for a system of nonlinear integral equations and functional equations in dynamic programming are also discussed.

2. Preliminaries

Following the structure of C^*-*avGMS* [9] and *mGMS* [66], a new space is introduced called C^*-algebra valued modular G-metric space (abbreviated C^*-*avmGMS*).

Definition 1. *Let Z be a nonempty set, and S_3 be the permutation group on $\{1, 2, 3\}$. A mapping $\Omega : (0, \infty) \times Z \times Z \times Z \to \mathbb{A}_+$ is called a C^*-avmGM on Z, if for any $z_1, b_2, c_3, d \in Z$ and $\alpha > 0$ it satisfies:*

(i) $\Omega_\alpha(z_1, b_2, c_3) = \theta$ *if* $z_1 = b_2 = c_3$,

(ii) $\Omega_\alpha(z_1, z_1, b_2) \succ \theta$, *for all* $z_1, b_2 \in Z$ *with* $z_1 \neq b_2$,

(iii) $\Omega_\alpha(z_{\sigma_1}, b_{\sigma_2}, c_{\sigma_3}) = \Omega_\alpha(z_1, b_2, c_3)$, $\sigma \in S_3$,

(iv) $\Omega_\alpha(z_1, z_1, b_2) \preceq \Omega_\alpha(z_1, b_2, c_3)$ *for all* $z_1, b_2, c_3 \in Z$ *with* $b_2 \neq c_3$,

(v) $\Omega_{\alpha + \mu}(z_1, b_2, c_3) \preceq \Omega_\alpha(z_1, d, d) + \Omega_\mu(d, b_2, c_3)$.

   Then $(Z, \mathbb{A}, \Omega)$ is said to be C^*-avmGMS.

Here we discuss some properties and definitions of C^*-*avmGMS*, as follows:

(a)   The essential property on a set Z of a C^*-*avmGM*, Ω is that for any $z_1, b_2, c_3 \in Z$ the function $0 < \alpha \to \Omega_\alpha(z_1, b_2, c_3) \in \mathbb{A}$ is nonincreasing on $(0, \infty)$. Moreover, if $0 < \mu < \alpha$, then

$$\Omega_\alpha(z_1, b_2, c_3) \preceq \Omega_{\alpha - \mu}(z_1, z_1, z_1) + \Omega_\mu(z_1, b_2, c_3)] = \Omega_\mu(z_1, b_2, c_3).$$

(b)   It can be easily checked as that, if $a_0 \in Z$ the set

$$Z_\Omega = \{a \in Z : \lim_{\alpha \to \infty} \Omega_\alpha(a, a_0, z) = \theta\} \text{ for some } z \in Z,$$

is a C^*-*avmGMS* with the generalized metric $G_\Omega^0 : Z_\Omega \times Z_\Omega \times Z_\Omega \to \mathbb{A}$ is given by

$$G_\Omega^0 = \inf\{\alpha > 0 : \|\Omega_\alpha(z_1, b_2, c_3)\| \leqslant \alpha\} \text{ for all } z_1, b_2, c_3 \in Z_\Omega,$$

called a C^*-*avmGMS*.

(c)   If $(Z, \mathbb{A}, \omega)$ is a C^*-*avmMS* then $(Z, \mathbb{A}, \omega)$ can define C^*-*avmGM* on Z by

$$(A^s)\ \Omega_\alpha^s(z_1, b_2, c_3) = \frac{1}{3}(\omega_\alpha(z_1, b_2) + \omega_\alpha(b_2, c_3) + \omega_\alpha(c_1, z_3)),$$

$$(B^m)\ \Omega_\alpha^m(z_1, b_2, c_3) = max(\omega_\alpha(z_1, b_2), \omega_\alpha(b_2, c_3), \omega_\alpha(c_1, z_3)), \text{ for all } \alpha > 0.$$

(d)   Any C^*-*avmGMS*, $(Z, \mathbb{A}, \Omega)$ induces a $C^* - avmM$, ω_α by

$$\omega_\alpha^\Omega(z_1, b_2) = \Omega_\alpha(z_1, z_1, b_2) + \Omega_\alpha(z_1, b_2, b_2), \text{ for all } \alpha > 0, \text{ and satisfies;}$$

$\Omega_\alpha(z_1, b_2, c_3) \leqslant \Omega_\alpha^s(z_1, b_2, c_3) \leqslant 2\Omega_\alpha(z_1, b_2, c_3)$, for all $\alpha > 0$, and $\frac{1}{2}\Omega_\alpha(z_1, b_2, c_3) \leqslant \Omega_\alpha^m(z_1, b_2, c_3) \leqslant 2\Omega_\alpha(z_1, b_2, c_3)$, for all $\alpha > 0$. Further, starting from a $C^* - avmM$, ω on Z, we have $\omega_\alpha^{\Omega^s}(z_1, b_2) = \frac{4}{3}\omega_\alpha(z_1, b_2)$ and $\omega_\alpha^{\Omega^m}(z_1, b_2) = 2\omega_\alpha(z_1, b_2)$.

Definition 2. *Let Z_Ω be a C^*-avmGMS. Then for each $\alpha > 0$,*

(1)   *Any sequence $\{a_n\}_{n \in \mathbb{N}}$ in Z_Ω is convergent to $a \in Z_\Omega$ with respect to $\mathbb{A}$ if, for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m \geqslant N$, $\Omega_\alpha(a, a_n, a_m) \prec \epsilon$.*
   Moreover, it is Ω-G-Cauchy if for all $n, m, l \geqslant N$, $\Omega_\alpha(a_n, a_m, a_l) \prec \epsilon$.

(2)   *A mapping T is Ω-G-continuous with respect to $\mathbb{A}$ in $B \subseteq Z_\Omega$ if for every sequence $\{a_n\}_{n \in \mathbb{N}} \subseteq B$ such that for all $n \geqslant N$, $\Omega_\alpha(a_n, z, z) \prec \epsilon$, then for all $n \geqslant N$, $\Omega_\alpha(Ta_n, Tz, Tz) \prec \epsilon$.*

(3) Z_Ω is Ω-complete if any Ω-G-Cauchy sequence with respect to $\mathbb{A}$ is Ω-G-convergent.

(4) A subset B of Z_Ω is Ω-G-bounded with respect to $\mathbb{A}$ if for each $\alpha > 0$ and $a_0 \in Z_\Omega$

$$\delta_\Omega(B) = sup\{\|\Omega_\alpha(a_0, a, b)\|; a, b \in B\} < \infty,$$

where, $\delta_\Omega(B)$ denotes the diameter of B in the C^*-avmGMS.

Proposition 1. *Let Z_Ω be a C^*-avmGMS, for each $\alpha > 0$. Then*

(1) $\{a_n\}_{n \in \mathbb{N}}$ *is a Ω-G-convergent to a with respect to $\mathbb{A}$;*

(2) $\Omega_\alpha(a_n, a_n, a) \to \theta$ *as $n \to \infty$;*

(3) $\Omega_\alpha(a_n, a, a) \to \theta$ *as $n \to \infty$; and*

(4) $\Omega_\alpha(a_n, a_m, a) \to \theta$ *as $n, m \to \infty$.*

are equivalent.

Proposition 2. *Let Z_Ω be a C^*-avmGMS, for each $\alpha > 0$. Then*

(1) $\{a_n\}_{n \in \mathbb{N}}$ *is a Ω-G-Cauchy with respect to $\mathbb{A}$; and*

(2) $\Omega_\alpha(a_m, a_n, a_n) \to \theta$ *as $n, m \to \infty$*

are equivalent.

Proposition 3. *Let $(Z, \mathbb{A}, \Omega)$ be a C^*-avmGMS. For any $z_1, b_2, c_3, a \in Z$, and $\alpha > 0$ it follows:*

(i) *if $\Omega_{2\alpha}(z_1, b_2, c_3) = \theta$ then $z_1 = b_2 = c_3$;*

(ii) $\Omega_{2\alpha}(z_1, b_2, c_3) \preceq \Omega_\alpha(z_1, z_1, b_2) + \Omega_\alpha(z_1, z_1, c_3)$;

(iii) $\Omega_{2\alpha}(z_1, b_2, b_2) \preceq 2\Omega_\alpha(b_2, z_1, z_1)$;

(iv) $\Omega_{2\alpha}(z_1, b_2, c_3) \preceq \Omega_\alpha(z_1, a, c_3) + \Omega_\alpha(a, b_2, c_3)$;

(v) $\Omega_{2\alpha}(z_1, b_2, c_3) \preceq \frac{2}{3}(\Omega_\alpha(z_1, b_2, a) + \Omega_\alpha(z_1, a, c_3) + \Omega_\alpha(a, b_2, c_3))$;

(vi) $\Omega_{4\alpha}(z_1, b_2, c_3) \preceq \Omega_\alpha(z_1, a, a) + \Omega_\alpha(b_2, a, a) + \Omega_\alpha(c_3, a, a)$.

Definition 3. *Let Z_Ω be a C^*-avmGMS, Ω_α is said to be symmetric if $\Omega_\alpha(z_1, z_1, b_2) = \Omega_\alpha(z_1, b_2, b_2)$ for all $z_1, b_2 \in Z_\Omega$ and $\alpha > 0$.*

Example 1. *Let $Z = \{0, 1, 2\}$ and consider, $\mathbb{A} = M_2(\mathbb{R})$. Let $C \in \mathbb{A}$ and $*$, be the involution map such that $C^* = C$, define $\|C\| = \sqrt{\sum_{i,j=1}^{2} |c_{ij}|^2}$. Clearly, $\mathbb{A}$ is a C^*-algebra. For $A = [a_{ij}]_{2 \times 2}$, $B = [b_{ij}]_{2 \times 2} \in M_2(\mathbb{R})$; we denote $A \preceq B$ if and only if $a_{ij} \leqslant b_{ij}$.*

Define $\Omega : (0, \infty) \times Z \times Z \times Z \to \mathbb{A}$ by $\Omega_\alpha(z_1, b_2, c_3) = diag\left(\left|\frac{g(z_1, b_2, c_3)}{\alpha}\right|, \left|\frac{g(z_1, b_2, c_3)}{\alpha}\right|\right)$, for all $z_1, b_2, c_3 \in Z$ and $\alpha > 0$, where

$$g(z_1, b_2, c_3) = \begin{cases} 0 \text{ if } z_1 = b_2 = c_3, \\ 1 \text{ if } (z_1, b_2, c_3) \in \{(0,0,1), (0,0,2), (1,1,2)\}, \\ 2 \text{ if } (z_1, b_2, c_3) \in \{(0,1,1), (0,2,2), (1,2,2), (0,1,2)\}. \end{cases}$$

Then it can be easily checked that Ω_α is a C^*-avmGMS. Since,

$$\Omega_\alpha(0, 0, 1) \neq \Omega_\alpha(0, 1, 1); \ \Omega_\alpha(0, 0, 2) \neq \Omega_\alpha(0, 2, 2); \ \Omega_\alpha(1, 1, 2) \neq \Omega_\alpha(1, 2, 2),$$

so Ω_α is not symmetric (see [52]).

Now if we take $g(z_1, b_2, c_3) = |z_1 - b_2| + |b_2 - c_3| + |c_3 - z_1|$, then it can be checked that Ω_α is a C^*-avmGMS and symmetric.

Example 2. *For the Lebesgue measurable set E and Hilbert space H let, $Z = L^\infty(E)$, $H = L^2(E)$ and $\mathbb{A} = B(H)$, the set of bounded linear operator on H. Define $\Omega : (0, \infty) \times Z \times Z \times Z \to B(H)_+$ by*

$$\Omega_\alpha(z_1, b_2, c_3) = \beta_{\left|\frac{z_1 - b_2}{\alpha}\right| + \left|\frac{b_2 - c_3}{\alpha}\right| + \left|\frac{c_3 - z_1}{\alpha}\right|} \ \forall \ z_1, b_2, c_3 \in Z, \ \alpha > 0,$$

where $\beta_\theta : H \to H$ is the multiplication operator defined by $\beta_\theta(\Phi) = \theta \cdot \Phi$, $\Phi \in H$. Then $(Z_\Omega, B(H), \Omega)$ is a Ω-G-complete C^*-avmGMS.

Example 3. *Let* $Z = l^\infty(S)$ *and* $H = l^2(S)$, $S \neq \phi$, *and* $\mathbb{A} = B(H)$. *Define* $\Omega : (0, \infty) \times Z \times Z \times Z \to B(H)_+$ *by*

$$\Omega_\alpha(\{f_n\}, \{g_n\}, \{l_n\}) = \beta_{\left| \frac{\{f_n\} - \{g_n\}}{\alpha} \right| + \left| \frac{\{g_n\} - \{l_n\}}{\alpha} \right| + \left| \frac{\{l_n\} - \{f_n\}}{\alpha} \right|} \ \forall \ \{f_n\}, \{g_n\}, \{l_n\} \in Z, \ \alpha > 0,$$

where β_θ *is described as in Example 2. Then* $(Z_\Omega, B(H), \Omega)$ *is a* Ω-G-complete C^*-avmGMS.

Example 4 (see [9]). *Let* $Z = \mathbb{R}$ *and* $\mathbb{A} = B(H)$. *Define* $\Omega : (0, \infty) \times Z \times Z \times Z \to B(H)_+$ *by*

$$\Omega_\alpha(z_1, b_2, c_3) = \left\{ \left| \frac{z_1 - b_2}{\alpha} \right| + \left| \frac{b_2 - c_3}{\alpha} \right| + \left| \frac{c_3 - z_1}{\alpha} \right| \right\} 1_{\mathbb{A}} \ \forall \ z_1, b_2, c_3 \in Z, \ \alpha > 0.$$

Then $(Z_\Omega, B(H), \Omega)$ *is a* Ω-G-complete C^*-avmGMS.

Definition 4 ([2,12]). *Define a continuous function* $H : \mathbb{A}_+ \times \mathbb{A}_+ \to \mathbb{A}$ *for* C^*-algebra $\mathbb{A}$. *If for any* $A, B \in \mathbb{A}_+$, *satisfies:*

(i) $H(A, B) \preceq A$; *and*
(ii) $H(A, B) = A \Rightarrow A = \theta$ *or* $B = \theta$.
       Then the function is called "C_*-class function".

Definition 5 ([2]). *A tripled* (ψ, φ, H_*) *where* $\psi : \mathbb{A}_+ \to \mathbb{A}_+$ *in* Ψ *(set of all continuous functions),* $\varphi : \mathbb{A}_+ \to \mathbb{A}_+$ *in* Φ *(the class of functions) and* $H_* : \mathbb{A}_+ \times \mathbb{A}_+ \to \mathbb{A}$. *If for any* $A, B \in \mathbb{A}_+$ *satisfies:*

$$A \preceq B \to H_*(\psi(A), \varphi(A)) \preceq H_*(\psi(B), \varphi(B)).$$

Then it is monotone.

Definition 6 ([63]). *Let* (Z, G) *be a G-metric space and* $T : Z \to Z$. *Then* T *is said to be* $G - \beta - \psi$-*contractive mapping of type A if there exist two functions* $\beta : Z \times Z \times Z \to [0, \infty)$ *and* $\psi \in \mathcal{F}_{com}^{(c)}$, *family of (c) comparision function such that*

$$\beta(z_1, z_2, Tz_1)G(Tz_1, Tz_2, T^2z_1) \leq \psi(G(z_1, z_2, Tz_1)), \ \forall \ z_1, z_2 \in Z.$$

According to Agarwal et al. [63] this type of contractive condition cannot be reduced to a quasi metric.

Asadi and Salimi [62] established the following theorem, which cannot be reduced to quasi metric space as well.

Theorem 1 ([62]). *Let* (Z, G) *be a G-metric space and* T *and* S *be two self-mappings on* Z. *For all* $z_1, z_2 \in Z$, *nondecreasing and continuous function* ψ *and lower semicontinuous function* ϕ; *if it satisfies* $\psi(G(Tz_1, STz_1, Tz_2)) \leq \psi(G(z_1, Sz_1, z_2)) - \phi(G(z_1, Sz_1, z_2))$. *Then* S *and* T *have a common unique fixed point.*

3. Main Results

Let $(Z_\Omega, \mathbb{A}, \Omega)$ be a Ω-complete C^*-avmGMS, and (T_1, T_2) be two self-mappings on Z_Ω, satisfying the conditions:

$$\psi(\|\Omega_\alpha(T_1a, T_1b, T_1b)\|1_{\mathbb{A}}) \preceq F_*(\psi(M(a, b, b)), \varphi(M(a, b, b))),$$

for which, $F_* \in C_*$, $\psi \in \Psi$ and $\varphi \in \Phi$ with strictly monotone (ψ, φ, F_*).

$$M(a,b,b) = \|q\|^2 \|\Omega_{2\alpha}(T_2a, T_2b, T_2b)\|1_{\mathbb{A}}$$
$$+ \|L\|^2 min\{\|\Omega_\alpha(T_1a, T_2a, T_2a)\|, \|\Omega_\alpha(T_1b, T_2a, T_2a)\|,$$
$$\|\Omega_\alpha(T_1a, T_2b, T_2b)\|, \|\Omega_\alpha(T_1b, T_2b, T_2b)\|\}1_{\mathbb{A}}.$$

In the setting $\Omega_\alpha(z_1, b_2, b_2) = \omega_\alpha(z_1, b_2)$ the contractive condition reduces to $\psi(\|\omega_\lambda(T_1a, T_1b)\|1_{\mathbb{A}}) \preceq F_*(\psi(M(a,b)), \varphi(M(a,b)))$, where

$$M(a,b,b) = M(a,b) = \|q\|^2 \|\omega_{2\alpha}(T_2a, T_2b)\|1_{\mathbb{A}}$$
$$+ \|L\|^2 min\{\|\omega_\alpha(T_1a, T_2a)\|, \|\omega_\alpha(T_1b, T_2a)\|,$$
$$\|\omega_\alpha(T_1a, T_2b)\|, \|\omega_\alpha(T_1b, T_2b)\|\}1_{\mathbb{A}}.$$

Proceeding as in ([59,60]) one can construct the same result in quasi C^*-*avmMS* instead of C^*-*avmGMS*.

Motivated by Theorem 1 [62], the following theorem's contractive condition cannot be reduced to quasi C^*-*avmMS*.

Theorem 2. *Let $(Z_\Omega, \mathbb{A}, \Omega)$ be a Ω-complete C^*-avmGMS, and T_1 and T_2 be two self-mappings on Z_Ω, satisfying the condition: for each $a, b \in Z_\Omega$, $q \in \mathbb{A}$ with $0 < \|q\| < 1$, and $\|\Omega_\alpha(T_1a, T_2T_1a, T_1b)\| < \infty$;*

$$\psi(\|\Omega_\alpha(T_1a, T_2T_1a, T_1b)\|1_{\mathbb{A}}) \preceq F_*(\psi(\|q\|^2 \|\Omega_{2\alpha}(a, T_2a, b)\|1_{\mathbb{A}}),$$
$$\varphi(\|q\|^2 \|\Omega_{2\alpha}(a, T_2a, b)\|1_{\mathbb{A}})),$$

for which, $F_* \in C_*$, $\psi \in \Psi$ and $\varphi \in \Phi$ with strictly monotone (ψ, φ, F_*). Then T_1 and T_2 have a common unique fixed point in Z_Ω.

Proof. Let $a_0 \in Z_\Omega$. Hence inductively $a_{n+1} = T_1a_n \ \forall n = 0, 1, 2, \dots$.

$$\psi(\|\Omega_\alpha(a_n, T_2a_n, a_n)\|1_{\mathbb{A}}) \preceq F_*(\psi(\|q\|^2 \|\Omega_{2\alpha}(a_{n-1}, T_2a_{n-1}, a_{n-1})\|1_{\mathbb{A}}),$$
$$\varphi(\|q\|^2 \|\Omega_{2\alpha}(a_{n-1}, T_2a_{n-1}, a_{n-1})\|1_{\mathbb{A}})),$$
$$\preceq \psi(\|q\|^2 \|\Omega_{2\alpha}(a_{n-1}, T_2a_{n-1}, a_{n-1})\|1_{\mathbb{A}}.$$

Since ψ is nondecreasing,

$$\|\Omega_\alpha(a_n, T_2a_n, a_n)\|1_{\mathbb{A}} \preceq \|q\|^2 \|\Omega_{2\alpha}(a_{n-1}, T_2a_{n-1}, a_{n-1})\|1_{\mathbb{A}},$$
$$\preceq \|q\|^2 \|\Omega_\alpha(a_{n-1}, T_2a_{n-1}, a_{n-1})\|1_{\mathbb{A}},$$
$$\dots,$$
$$\preceq \|q\|^{2n} \|\Omega_\alpha(a_0, T_2a_0, a_0)\|1_{\mathbb{A}},$$
$$\to 0 \text{ as } n \to \infty \text{ (since } \|q\| < 1).$$

Now we show that $\{a_n\}_{n=0}^\infty$ and $\{T_2a_n\}_{n=0}^\infty$ are Ω-G-Cauchy sequence. Suppose there exist $\epsilon > 0$ and subsequence $\{a_{m(k)}\}$ and $\{a_{n(k)}\}$ with $n(k) > m(k) > k > 0$ such that

$$\|\Omega_\alpha(a_{n(k)}, a_{m(k)}, a_{m(k)})\|1_{\mathbb{A}} \preceq \|\Omega_{\frac{\alpha}{2}}(a_{n(k)}, T_2a_{m(k)}, a_{m(k)})\|1_{\mathbb{A}} + 2\|\Omega_{\frac{\alpha}{4}}(T_2a_{m(k)}, a_{m(k)}, a_{m(k)})\|1_{\mathbb{A}}.$$

Hence, $\lim_{k\to\infty} \Omega_\alpha(a_{n(k)}, a_{m(k)}, a_{m(k)}) = \theta$. Therefore $\{a_n\}_{n=0}^\infty$ is a Ω-G-Cauchy sequence and $\lim_{n\to\infty} \Omega_\alpha(a_n, l, l) = \theta$, for all $\alpha > 0$ and for some $l \in Z_\Omega$. Suppose there exist $\epsilon > 0$ and subsequence $\{T_2 a_{m(k)}\}$ and $\{T_2 a_{n(k)}\}$ with $n(k) > m(k) > k > 0$ such that

$$\|\Omega_\alpha(T_2 a_{n(k)}, T_2 a_{m(k)}, T_2 a_{m(k)})\|1_\mathbb{A} \preceq \|\Omega_{\frac{\alpha}{2}}(a_{n(k)}, T_2 a_{n(k)}, a_{n(k)})\|1_\mathbb{A}$$
$$+ 2\|\Omega_{\frac{\alpha}{4}}(T_2 a_{m(k)}, a_{n(k)}, a_{n(k)})\|1_\mathbb{A}.$$

Hence, $\lim_{k\to\infty} \Omega_\alpha(T_2 a_{n(k)}, T_2 a_{m(k)}, T_2 a_{m(k)}) = \theta$. Therefore $\{T_2 a_n\}_{n=0}^\infty$ is a Ω-G-Cauchy sequence and $\lim_{n\to\infty} \Omega_\alpha(T_2 a_n, T_2 p, T_2 p) = \theta$, for all $\alpha > 0$ and for some $p \in Z_\Omega$.
Since $\|\Omega_\alpha(a_n, T_2 a_n, a_n)\|1_\mathbb{A} \to 0$ as $n \to \infty$, so $l = T_2 p$.
Now,

$$\psi(\|\Omega_\alpha(a_n, T_2 a_n, T_1 l)\|1_\mathbb{A}) = \psi(\|\Omega_\alpha(T_1 a_{n-1}, T_2 T_1 a_{n-1}, T_1 l)\|1_\mathbb{A}),$$
$$\preceq F_*(\psi(\|q\|^2\|\Omega_\alpha(a_{n-1}, T_2 a_{n-1}, l)\|1_\mathbb{A}),$$
$$\varphi(\|q\|^2\|\Omega_\alpha(a_{n-1}, T_2 a_{n-1}, l)\|1_\mathbb{A})),$$
$$\preceq \psi(\|q\|^2\|\Omega_\alpha(a_{n-1}, T_2 a_{n-1}, l)\|1_\mathbb{A}).$$

Since ψ is non decreasing,

$$\|\Omega_\alpha(a_n, T_2 a_n, T_1 l)\|1_\mathbb{A} \preceq \|q\|^{2n}\|\Omega_\alpha(a_0, T_2 a_0, l)\|1_\mathbb{A},$$
$$\to 0 \text{ as } n \to \infty \text{ (since } \|q\| < 1).$$

Hence $l = T_1 l = T_2 p$.

$$\psi(\|\Omega_\alpha(l, T_2 l, l)\|1_\mathbb{A}) = \psi(\|\Omega_\alpha(T_1 l, T_2 T_1 l, T_1 l)\|1_\mathbb{A}),$$
$$\preceq F_*(\psi(\|q\|^2\|\Omega_\alpha(l, T_2 l, l)\|1_\mathbb{A}), \varphi(\|q\|^2\|\Omega_\alpha(l, T_2 l, l)\|1_\mathbb{A})),$$
$$\preceq \psi(\|q\|^2\|\Omega_\alpha(l, T_2 l, l)\|1_\mathbb{A}).$$

Since ψ is non decreasing and $\|q\| < 1$, so $\Omega_\alpha(l, T_2 l, l) = \theta$. Hence $l = T_1 l = T_2 l$.
Uniqueness:
If possible, let $r \in Z_\Omega$ such that $r = T_1 r = T_2 r$.

$$\psi(\|\Omega_\alpha(l, T_2 l, r)\|1_\mathbb{A}) = \psi(\|\Omega_\alpha(T_1 l, T_2 T_1 l, T_1 r)\|1_\mathbb{A}),$$
$$\preceq F_*(\psi(\|q\|^2\|\Omega_\alpha(l, T_2 l, r)\|1_\mathbb{A}), \varphi(\|q\|^2\|\Omega_\alpha(l, T_2 l, r)\|1_\mathbb{A})),$$
$$\preceq \psi(\|q\|^2\|\Omega_\alpha(l, T_2 l, r)\|1_\mathbb{A}).$$

Clearly, $l = r$. Hence T_1 and T_2 have a common unique fixed point r. $\quad\square$

Corollary 1. *Let $(Z_\Omega, \mathbb{A}, \Omega)$ be a Ω-complete C^*-avmGMS, and T be a self-mapping on Z_Ω, satisfying the condition: for each $a, b \in Z_\Omega, q \in \mathbb{A}$ with $0 < \|q\| < 1$, and $\|\Omega_\alpha(Ta, T^2 a, Tb)\| < \infty$;*

$$\psi(\|\Omega_\alpha(Ta, T^2 a, Tb)\|1_\mathbb{A}) \preceq F_*(\psi(\|q\|^2\|\Omega_{2\alpha}(a, Ta, b)\|1_\mathbb{A}),$$
$$\varphi(\|q\|^2\|\Omega_{2\alpha}(a, Ta, b)\|1_\mathbb{A})),$$

for which, $F_* \in C_*$, $\psi \in \Psi$ and $\varphi \in \Phi$ with strictly monotone (ψ, φ, F_*). Then T has a unique fixed point in Z_Ω.

Example 5. *As in Example 2, let $Z = l^\infty(S)$, $H = l^2(S)$, and $\mathbb{A} = B(H)$. Define $\Omega : (0, \infty) \times Z \times Z \times Z \to B(H)_+$ by*

$$\Omega_\alpha(\{f_m\}, \{g_n\}, \{h_l\}) = \beta_{\left|\frac{\{f_m\} - \{g_n\}}{\alpha}\right| + \left|\frac{\{g_n\} - \{h_l\}}{\alpha}\right| + \left|\frac{\{h_l\} - \{f_m\}}{\alpha}\right|} \quad \forall \{f_m\}, \{g_n\}, \{h_l\} \in Z;$$

$\alpha > 0, m, n, l \in \mathbb{N}$. Then $(Z_\Omega, B(H), \Omega)$ is a Ω-G-complete C^*-avmGMS.

Proof. Define $T_1 : Z_\Omega \rightarrow Z_\Omega$ by $T_1(\{f_m\}) = \{\left(\frac{m+1}{m}\right)^2, 0, 0, \ldots\}$ where $\{f_m\} = \{\frac{m+1}{m}, 0, 0, \ldots\} \in Z_\Omega$, $m \in \mathbb{N}$. (see [70]) Again define $T_2 : Z_\Omega \rightarrow Z_\Omega$ by $T_2(\{g_m\}) = \{\frac{m+1}{m}, 0, 0, \ldots\}$ where $\{g_m\} = \{(\frac{m+1}{m})^2, 0, 0, \ldots\} \in Z_\Omega$, $m \in \mathbb{N}$.

Suppose ψ and φ are two self-mappings on $\mathbb{A}_+$ such that $\psi(A) = 2A$ and $\varphi(A) = A$ for all $A \in \mathbb{A}_+$, and

$$\begin{cases} F_* : \mathbb{A}_+ \times \mathbb{A}_+ \rightarrow \mathbb{A}, \\ F_*(A, B) = A - B. \end{cases}$$

Then (ψ, φ, F_*) is strictly monotone. For all $\{f_m\}, \{f_n\} \in Z_\Omega = l^\infty(S)$, and $\alpha > 0$ we have clearly,

$$\|\Omega_\alpha(T_1(\{f_m\}), T_2 T_1(\{f_m\}), T_1(\{f_n\}))\| < \infty; \ n, m \in \mathbb{N}(m \geqslant n).$$

For every $q, L \in \mathbb{A}$ with $0 < \|q\| < 1$, $\|L\| \neq 0$, we have

$$\psi(\|\Omega_\alpha(T_1\{f_m\}, T_2 T_1\{f_m\}, T_1\{f_n\})\|1_\mathbb{A}) \preceq F_*(\psi(\|q\|^2\|\{f_m\}, T_2\{f_m\}, \{f_n\}\|),$$
$$\varphi(\|q\|^2\|\{f_m\}, T_2\{f_m\}, \{f_n\}\|).$$

Hence it satisfies all the conditions of Theorem 2. So, T_1 and T_2 have common unique fixed points $\{1, 0, 0, \ldots\}$. $\square$

4. Ulam–Hyers Stability Results in C^*-*avmGMS*

Let $(Z, \mathbb{A}, \Omega)$ be a C^*-*avmGMS* and $T : Z_\Omega \rightarrow Z_\Omega$ be a mapping. Let set of n fixed point equations be

$$T^n z = z, n \in \mathbb{N} \tag{1}$$

called a generalized Ulam–Hyers stability if

(i)   there exists a increasing mapping $\xi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with continuity at 0 and $\xi(0) = 0$;
(ii)   for any $\epsilon > 0$ and for each $z^* \in Z_\Omega$ an ϵ solution of the Equation (1), which satisfies

$$\Omega_\alpha(z^*, T^n z^*, Tz^*) \preceq \epsilon, \ n \in \mathbb{N}. \tag{2}$$

There exists a solution $x^* \in Z_\Omega$ of (1) such that $\Omega_\alpha(z^*, x^*, x^*) \preceq \xi(\epsilon)$. Then for any $s \in \mathbb{R}_+$ and $k > 0$; $\xi(s) = k.s$ implies Ulam–Hyers stability of (1).

In the following theorem, two fixed point equations, $Tz = z$ and $T^2 z = z$ are considered.

Theorem 3. *Let $(Z_\Omega, \mathbb{A}, \Omega)$ be a Ω-complete C^*-avmGMS satisfying all the condition of Corollary 1. Moreover,*

(a) $\psi(\|\Omega_\alpha(Ta, T^2a, Tb)\|1_\mathbb{A} + K) \preceq F_*(\psi(\|q\|^2\|\Omega_{2\alpha}(a, Ta, b)\|1_\mathbb{A} + K), \varphi(\|q\|^2\|\Omega_{2\alpha}(a, Ta, b)\|1_\mathbb{A} + K), K \in \mathbb{A}_+.$
(b) $\|\Omega_\alpha(Tz^*, Tz^*, z^*)\|1_\mathbb{A} \preceq (1 - \|q\|^2)\gamma 1_\mathbb{A}, \ \forall \alpha > 0 ; \gamma > 0.$

Then Equation (1) is Ulam–Hyers stable.

Proof. From Corollary 1 $Fix(T) = \{x^*\}$. Let $\epsilon > 0$ and $z^* \in Z_\Omega$ be a solution of Equation (2)

$$\psi(\|\Omega_\alpha(x^*, x^*, z^*)\|1_\mathbb{A}) = \psi(\|\Omega_\alpha(x^*, Tx^*, z^*)\|1_\mathbb{A}) = \psi(\|\Omega_\alpha(Tx^*, T^2x^*, z^*)\|1_\mathbb{A}),$$
$$\preceq \psi(\|\Omega_{\frac{\alpha}{2}}(Tx^*, T^2x^*, Tz^*)\|1_\mathbb{A} + \|\Omega_{\frac{\alpha}{2}}(Tz^*, Tz^*, z^*)\|1_\mathbb{A}),$$
$$\preceq \psi(\|\Omega_{\frac{\alpha}{2}}(Tx^*, T^2x^*, Tz^*)\|1_\mathbb{A} + (1 - \|q\|^2)\gamma 1_\mathbb{A}),$$
$$\preceq F_*(\psi(\|q\|^2\|\Omega_\alpha(x^*, Tx^*, z^*)\|1_\mathbb{A} + (1 - \|q\|^2)\gamma 1_\mathbb{A}),$$
$$\varphi(\|q\|^2\|\Omega_\alpha(x^*, Tx^*, z^*)\|1_\mathbb{A} + (1 - \|q\|^2)\gamma 1_\mathbb{A})),$$
$$\preceq \psi(\|q\|^2\|\Omega_\alpha(x^*, Tx^*, z^*)\|1_\mathbb{A} + (1 - \|q\|^2)\gamma 1_\mathbb{A}),$$

this implies that $\|\Omega_\alpha(x^*, x^*, z^*)\|1_\mathbb{A} \preceq \gamma 1_\mathbb{A}$. Hence Equation (1) is Ulam–Hyers stable. □

Theorem 4. *Let $(Z_\Omega, \mathbb{A}, \Omega)$ be a Ω-complete C^*-avmGMS satisfying all the conditions of Theorem 3 with (a). Moreover, the onto function $\pi : [0, \infty) \to [0, \infty)$ such that $\pi(r) = \lambda r$ is strictly increasing. Then*

(a)   Equation (1) is generalized Ulam–Hyers stable.

(b) $Fix(T) = \{x^*\}$ and if $\{z_n\} \in Z_\Omega, n \in \mathbb{N}$ are such that $\|\Omega_\alpha(z_n, Tz_n, Tz_n)\|1_\mathbb{A} \to 0$ as $n \to \infty$, then $z_n \to x^*$ as $n \to \infty$. (well-posed)

(c)   If $S : Z_\Omega \to Z_\Omega$ such that $\|\Omega_\alpha(Tz, Tz, Sz)\|1_\mathbb{A} \preceq \eta 1_\mathbb{A}, \ \forall z \in Z_\Omega \ \eta \in [0, \infty)$ then $p^* \in Fix(S) \Rightarrow \|\Omega_\alpha(x^*, p^*)\|1_\mathbb{A} \preceq \pi^{-1}(\frac{\alpha}{1-\|q\|^2\|}\eta 1_\mathbb{A}).$

Proof. (a) Let $Fix(T) = \{x^*\}, \epsilon > 0$ and $z^* \in Z_\Omega$.

$$\psi(\|\Omega_\alpha(x^*, x^*, z^*)\|1_\mathbb{A}) = \psi(\|\Omega_\alpha(x^*, Tx^*, z^*)\|1_\mathbb{A}) = \psi(\|\Omega_\alpha(Tx^*, T^2x^*, z^*)\|1_\mathbb{A}),$$
$$\preceq \psi(\|\Omega_{\frac{\alpha}{2}}(Tx^*, T^2x^*, Tz^*)\|1_\mathbb{A} + \|\Omega_{\frac{\alpha}{2}}(Tz^*, Tz^*, z^*)\|1_\mathbb{A}),$$
$$\preceq \psi(\|\Omega_{\frac{\alpha}{2}}(Tx^*, T^2x^*, Tz^*)\|1_\mathbb{A} + \epsilon 1_\mathbb{A}),$$
$$\preceq F_*(\psi(\|q\|^2\|\Omega_\alpha(x^*, Tx^*, z^*)\|1_\mathbb{A} + \epsilon 1_\mathbb{A}),$$
$$\varphi(\|q\|^2\|\Omega_\alpha(x^*, Tx^*, z^*)\|1_\mathbb{A} + \epsilon 1_\mathbb{A})),$$
$$\preceq \psi(\|q\|^2\|\Omega_\alpha(x^*, Tx^*, z^*)\|1_\mathbb{A} + \epsilon 1_\mathbb{A}),$$

this implies that $\|\Omega_\alpha(x^*, x^*, z^*)\|1_\mathbb{A} \preceq \frac{1}{1-\|q\|^2\|}\epsilon 1_\mathbb{A}$. So,

$$\pi(\|\Omega_\alpha(x^*, x^*, z^*)\|1_\mathbb{A}) = \lambda\|\Omega_\alpha(x^*, x^*, z^*)\|1_\mathbb{A} \preceq \frac{\lambda}{1 - \|q\|^2\|}\epsilon 1_\mathbb{A}.$$

Therefore we have, $\|\Omega_\alpha(x^*, x^*, z^*)\|1_\mathbb{A} \preceq \pi^{-1}(\frac{\lambda}{1-\|q\|^2\|}\epsilon 1_\mathbb{A})$. Hence, Equation (1) is generalized Ulam–Hyers stable.

(b) Let $Fix(T) = \{x^*\}, \epsilon > 0$ and $\{z_n\} \in Z_\Omega$.

$$\psi(\|\Omega_\alpha(x^*, Tx^*, z_n)\|1_\mathbb{A}) \preceq \psi(\|\Omega_{\frac{\alpha}{2}}(z_n, Tz_n, Tz_n)\|1_\mathbb{A} + \|\Omega_{\frac{\alpha}{2}}(Tx^*, T^2x^*, Tz_n)\|1_\mathbb{A}),$$
$$\preceq F_*(\psi(\|q\|^2\|\Omega_\alpha(x^*, Tx^*, z_n)\|1_\mathbb{A} + \|\Omega_{\frac{\alpha}{2}}(z_n, Tz_n, Tz_n)\|1_\mathbb{A}),$$
$$\varphi(\|q\|^2\|\Omega_\alpha(x^*, Tx^*, z_n)\|1_\mathbb{A} + \|\Omega_{\frac{\alpha}{2}}(z_n, Tz_n, Tz_n)\|1_\mathbb{A})),$$
$$\preceq \psi(\|q\|^2\|\Omega_\alpha(x^*, Tx^*, z_n)\|1_\mathbb{A} + \|\Omega_{\frac{\alpha}{2}}(z_n, Tz_n, Tz_n)\|1_\mathbb{A}),$$

this implies that

$$\|\Omega_\alpha(x^*, x^*, z_n)\|1_\mathbb{A} \preceq \frac{1}{1 - \|q\|^2\|}\|\Omega_{\frac{\alpha}{2}}(z_n, Tz_n, Tz_n)\|1_\mathbb{A},$$
$$\to 0 \ as \ n \to \infty.$$

Hence, we have $z_n \to x^*$ *as* $n \to \infty$.
(c) Let $Fix(T) = \{x^*\}$ and $p^* \in Fix(S)$.

$$\begin{aligned}
\psi(\|\Omega_\alpha(x^*, Tx^*, p^*)\|1_\mathbb{A}) &= \psi(\|\Omega_\alpha(Tx^*, T^2x^*, p^*)\|1_\mathbb{A}), \\
&\preceq \psi(\|\Omega_{\frac{\alpha}{2}}(Tx^*, T^2x^*, Tp^*)\|1_\mathbb{A} + \|\Omega_{\frac{\alpha}{2}}(Tp^*, Tp^*, p^*)\|1_\mathbb{A}), \\
&\preceq \psi(\|\Omega_{\frac{\alpha}{2}}(Tx^*, T^2x^*, Tp^*)\|1_\mathbb{A} + \|\Omega_{\frac{\alpha}{2}}(Tp^*, Tp^*, Sp^*)\|1_\mathbb{A}), \\
&\preceq F_*(\psi(\|q\|^2\|\Omega_\alpha(x^*, Tx^*, p^*)\|1_\mathbb{A} + \eta 1_\mathbb{A}), \\
&\qquad \varphi(\|q\|^2\|\Omega_\alpha(x^*, Tx^*, p^*)\|1_\mathbb{A} + \eta 1_\mathbb{A})), \\
&\preceq \psi(\|q\|^2\|\Omega_\alpha(x^*, Tx^*, p^*)\|1_\mathbb{A} + \eta 1_\mathbb{A}),
\end{aligned}$$

this implies that $\|\Omega_\alpha(x^*, x^*, p^*)\|1_\mathbb{A} \preceq \frac{1}{1-\|q\|^2}\eta 1_\mathbb{A}$. So,

$$\pi(\|\Omega_\alpha(x^*, x^*, p^*)\|1_\mathbb{A}) = \lambda\|\Omega_\alpha(x^*, x^*, p^*)\|1_\mathbb{A} \preceq \frac{\alpha}{1 - \|q\|^2}\eta 1_\mathbb{A}.$$

Therefore we have, $\|\Omega_\alpha(x^*, x^*, p^*)\|1_\mathbb{A} \preceq \pi^{-1}(\frac{\lambda}{1-\|q\|^2}\eta 1_\mathbb{A})$. $\square$

5. Applications

Shen et al. [9] provided an application for a type of differential equation in C^*-*avGMS*. Pathak et al. [71] for common fixed point and Moeini et al. [2] for C^*-*avmGMS* provided applications to nonlinear integral equations. All of the above inspired the following application (see also [13,14,23,37,56,68,69,72–74]).

First remind, as in Example 2, for all $\alpha > 0$ and $f, g, h \in L^\infty(E)$, define $\Omega : (0, \infty) \times L^\infty(E) \times L^\infty(E) \times L^\infty(E) \to B(L^2(E))_+$ by

$$\Omega_\alpha(f, g, h) = \beta_{\left|\frac{f-g}{\alpha}\right| + \left|\frac{g-h}{\alpha}\right| + \left|\frac{h-f}{\alpha}\right|}.$$

Then $(L^\infty(E)_\Omega, B(L^2(E)), \Omega)$ is a Ω-G-complete C^*-*avmGMS*.

Theorem 5. *Consider the following system of nonlinear integral equations:*

$$z(r) = I(r) + v(r, z(r)) + \mu \int_E t(r, s)q(s, z(s))ds, \tag{3}$$

where $r, s \in E$; $\mu \in \mathbb{F}$; $z, I \in L^\infty(E)_\Omega$ and $v(r, z(r)), t(r, s), q(s, z(s))$ are all real or complex valued functions, which are measurable in r and s on E. Suppose

(i) $\sup_{s \in E} \int_E |t(r, s)|dr = M < +\infty$;
(ii) *For all $s \in E$; $z_1, z_2 \in L^\infty(E)_\Omega$ and $v(s, z_1(s)), v(s, z_2(s)) \in L^\infty(E)_\Omega$, there exists $N > 1$ such that*
$\frac{|v(s, z_1(s) - v(s, z_2(s)|}{2\sqrt{2}} \geq N|z_1(s) - z_2(s)|$;
(iii) *For all $s \in E$; $z_1, z_2 \in L^\infty(E)_\Omega$ and $q(s, z_1(s), q(s, z_2(s) \in L^\infty(E)_\Omega$, there exists $L > 0$ such that $|q(s, z_1(s) - q(s, z_2(s)| \leq L|z_1(s) - z_2(s)|$;*
(iv) *For a nonempty set $\mathcal{B}$ consists of $f, g \in L^\infty(E)_\Omega$ and $K : [0, 1] \times [0, 1] \times \mathbb{R}^+ \to \mathbb{R}^+$, such that $v(r, g(r)) = g(r) - I(r) - \alpha \int_E t(r, s)q(s, g(s))ds = f(r)$, and $|I(r) + \alpha \int_E t(r, s) q(s, f(r) - I(r) - \mu \int_E t(r, s)q(s, f(s))ds)ds| \leq \frac{1+|\mu|ML}{2\sqrt{2}N}|f(r) - v(r, f(r))|$.*

Then Equation (3) has a unique solution for each $\mu \in \mathbb{F}$ with $\frac{1+|\mu|ML}{N} < 1$.

Proof. Define $T_1, T_2 : Z_\Omega \to Z_\Omega$ by

$$T_1 z(r) = z(r) - I(r) - \mu \int_E t(r,s)q(s,z(s))ds,$$

$$T_2 z(r) = v(r, z(r)).$$

Set $\|q\|^2 = \frac{1+|\mu|ML}{N} (< 1)$. Let ψ and φ be two self-mappings on $B(L^2(E))_+$ such that $\psi(A_1) = \frac{1}{2}A_1$ and $\varphi(A_2) = \frac{1}{4}A_2$ for all $A_1, A_2 \in B(L^2(E))_+$, and

$$\begin{cases} F_* : B(L^2(E))_+ \times B(L^2(E))_+ \to B(L^2(E)), \\ F_*(A_1, A_2) = \frac{1}{\sqrt{2}}A_1. \end{cases}$$

Clearly, (ψ, φ, F_*) is strictly monotonic. Let $f, g \in \mathcal{B}$.

$$\psi(\|\Omega_\alpha(T_1 f, T_2 T_1 f, T_1 g)\|1_\mathbb{A}) = \frac{1}{2}\|\Omega_\alpha(T_1 f, T_2 T_1 f, T_1 g)\|1_\mathbb{A},$$

$$= \frac{1}{2}\|\beta_{|\frac{T_1 f - T_2 T_1 f}{\alpha}| + |\frac{T_2 T_1 f - T_1 g}{\alpha}| + |\frac{T_1 g - T_1 f}{\alpha}|}\|1_\mathbb{A},$$

$$= \frac{1}{2}\|\frac{T_1 f - T_2 T_1 f}{\alpha}\|_\infty 1_\mathbb{A} + \|\frac{T_2 T_1 f - T_1 g}{\alpha}\|_\infty 1_\mathbb{A} + \|\frac{T_1 f - T_1 g}{\alpha}\|_\infty 1_\mathbb{A},$$

$$= \|\frac{T_1 f - T_2 T_1 f}{\alpha}\|_\infty 1_\mathbb{A} + \frac{1}{\alpha}\sup_{s \in E}\left|(f-g) + \mu \int_E t(r,s)(q(s,g(s)) - q(s,f(s)))ds\right|1_\mathbb{A},$$

$$\preceq \|\frac{T_1 f - T_2 T_1 f}{\alpha}\|_\infty 1_\mathbb{A} + \frac{1+|\mu|ML}{\alpha}\|f-g\|_\infty 1_\mathbb{A},$$

$$\preceq \|\frac{T_1 f - T_2 T_1 f}{\alpha}\|_\infty 1_\mathbb{A} + \frac{1+|\mu|ML}{2\sqrt{2}N}\|\frac{v(r,f(r)) - v(r,g(r))}{\alpha}\|_\infty 1_\mathbb{A},$$

$$\preceq \|\frac{I(r) + \alpha \int_E t(r,s)q(s,f(r) - I(r) - \mu \int_E t(r,s)q(s,f(s))ds)ds}{\alpha}\|_\infty 1_\mathbb{A}$$

$$+ \frac{1+|\mu|ML}{2\sqrt{2}N}\|\frac{T_2 f - T_2 g}{\alpha}\|_\infty 1_\mathbb{A},$$

$$\preceq \frac{1}{2\sqrt{2}}\|q\|^2(\|\frac{f - T_2 f}{\alpha}\|_\infty 1_\mathbb{A} + \|\frac{T_2 f - g}{\alpha}\|_\infty 1_\mathbb{A} + \|\frac{g - f}{\alpha}\|_\infty 1_\mathbb{A}),$$

$$\preceq \frac{1}{2\sqrt{2}}\|q\|^2\|\Omega_\alpha(f, T_2 f, g)\|1_\mathbb{A},$$

$$\preceq F_*(\psi(\|q\|^2\|\Omega_\alpha(f, T_2 f, g)\|1_\mathbb{A}), \phi(\|q\|^2\|\Omega_\alpha(f, T_2 f, g)\|1_\mathbb{A})).$$

By Theorem 2, we have a unique solution to the nonlinear integral Equation (3). $\square$

Let Z_Ω be a C^*-*avmGMS* and Q be a Banach space and $P \subseteq Q$. Define $\Theta : V \times P \to V$ and $D : V \times P \times \mathbb{R} \to \mathbb{R}$ be two functions, where $V \subseteq Z_\Omega$. Let $B(V)$ be the set of Banach spaces consisting all real functional on V. Define a norm, $\|a\| = \sup_{r \in V}|a(r)|$ and consider the functional equation arising in dynamic programming ([75,76])

$$a(r) = \sup_{s \in P}\{D(r, s, a(\Theta(r,s)))\} \tag{4}$$

where $a \in B(V)$. Define a C^*-*avmGMS* on $B(V)$ as in Example 4 by

$$\Omega_\alpha(f, g, h) = \left\{\left|\frac{f(r) - g(r)}{\alpha}\right| + \left|\frac{g(r) - h(r)}{\alpha}\right| + \left|\frac{h(r) - f(r)}{\alpha}\right|\right\}1_\mathbb{A},$$

for all $\alpha > 0$ and $f, g, h \in B(V)$.

Theorem 6. *Let T be a self-mapping on $B(V)$, defined by $Tf(r) = \sup_{s \in P}\{D(r, s, f(\Theta(r, s)))\}$, and $T^2 f(r) = \sup_{s \in P}\{D(r, s, Tf(\Theta(r, s)))\}$. If,*

$$|D(r, s, f(\Theta(r, s))) - D(r, s, g(\Theta(r, s)))| \leq \frac{\|q\|^2}{2}|f(s) - g(s)|1_{\mathbb{A}},$$

$$|D(r, s, Tf(\Theta(r, s))) - D(r, s, g(\Theta(r, s)))| \leq \frac{\|q\|^2}{2}|Tf(s) - g(s)|1_{\mathbb{A}},$$

$$|D(r, s, Tf(\Theta(r, s))) - D(r, s, f(\Theta(r, s)))| \leq \frac{\|q\|^2}{2}|Tf(s) - f(s)|1_{\mathbb{A}}, \ q < 1.$$

Then Equation (4) has unique bounded solution.

Proof. Let $r \in V$ and $f(r) \in B(V)$. Then there exists $s_1, s_2 \in P$ and $\epsilon > 0$ such that

$$Tf(r) \prec D(r, s_1, f(\Theta(r, s_1))) + \epsilon, \tag{5}$$
$$T^2 f(r) \prec D(r, s_1, Tf(\Theta(r, s_1))) + \epsilon, \tag{6}$$
$$Tg(r) \prec D(r, s_2, g(\Theta(r, s-2))) + \epsilon, \tag{7}$$
$$Tf(r) \preceq D(r, s_2, f(\Theta(r, s_2))), \tag{8}$$
$$T^2 f(r) \preceq D(r, s_2, Tf(\Theta(r, s_2))), \tag{9}$$
$$Tg(r) \preceq D(r, s_1, g(\Theta(r, s_1))). \tag{10}$$

From (5) and (10) we have

$$\begin{aligned} Tf(r) - Tg(r) &\prec D(r, s_1, f(\Theta(r, s_1))) - D(r, s_1, g(\Theta(r, s_1))) + \epsilon, \\ &\leq |D(r, s_1, f(\Theta(r, s_1))) - D(r, s_1, g(\Theta(r, s_1)))| + \epsilon, \\ &\leq \frac{\|q\|^2}{2}|f(s) - g(s)|1_{\mathbb{A}} + \epsilon. \end{aligned}$$

Again, from (7) and (8)

$$Tg(r) - Tf(r) \leq \frac{\|q\|^2}{2}|f(s) - g(s)|1_{\mathbb{A}} + \epsilon.$$

We can write

$$\left|\frac{Tf(r) - Tg(r)}{\alpha}\right|1_{\mathbb{A}} \leq \|q\|^2 \left|\frac{f(s) - g(s)}{2\alpha}\right|1_{\mathbb{A}}, \tag{11}$$

$$\left|\frac{T^2 f(r) - Tf(r)}{\alpha}\right|1_{\mathbb{A}} \leq \|q\|^2 \left|\frac{Tf(s) - T(s)}{2\alpha}\right|1_{\mathbb{A}}, \tag{12}$$

$$\left|\frac{T^2 f(r) - Tg(r)}{\alpha}\right|1_{\mathbb{A}} \leq \|q\|^2 \left|\frac{Tf(s) - g(s)}{2\alpha}\right|1_{\mathbb{A}}. \tag{13}$$

Clearly, from (11)–(13) we get, $\Omega_\alpha(Tf, T^2 f, Tg) \preceq \|q\|^2 \Omega_{2\alpha}(f, Tf, g)$. So from Corollary 1, we can conclude that the functional equation has a unique solution (4). □

6. Conclusions

In this paper, we introduce C^*-*avmGMS* with some properties and examples. Using "C_*-class function" we studied some fixed point results. To validate the results, we provided some examples and applications. The stability of a fixed point result is checked by Ulam–Hyers stability. The following are some of the study's most significant observations:

(i) It is observed that all the results in G-metric spaces, modular G-metric spaces, C^*-*avGMS*, and C^*-*avmGMS* cannot be obtained directly in the setting of quasi metric of these spaces.

(ii)   The results produced in C^*-*avmGMS* extend and generalize certain previous findings in the literature.

(iii) Applications in nonlinear integral equations and functional equations in dynamic programming of the space C^*-*avmGMS* and examples in C^*-*avmGMS* pave the way for a realistic result.

(iv)   The defined Ulam–Hyers stability for C^*-*avmGMS* is used to check the stability problem of fixed point equations, and can also be used to check stability for fixed point equations in *G*-metric spaces, modular *G*-metric spaces, and C^*-*avGMS*, respectively.

(v)   The results in C^*-*avmGMS* can be used to study a wide range of nonlinear problems.

Limitation and Future Perspectives: If the contractivity condition of the fixed point result on a C^*-*avmGMS* can be simplified to two variables then an analogous fixed point result in the context of C^*-*avGMS* can be established easily. Some applications of C^*-*avmGMS* may include differential equations, entropy analysis, integral equations, integrodifferential equations, noncommutative geometry, functional equations, quantum mechanics, string theory, etc. The presented results might actually be helpful to researchers in the literature of fixed point theory and further investigation into different generalized modular metric spaces and generalized metric spaces in the setting of C^*-algebra.

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Abbreviations

The following abbreviations are used in this manuscript:

mGMS	Modular *G*-Metric Spaces
C^*-*avMS*	C^*-algebra valued Metric Spaces
C^*-*avb-MS*	C^*-algebra valued *b*-Metric Spaces
C^*-*avSMS*	C^*-algebra valued *S*-Metric Spaces
C^*-*avGMS*	C^*-algebra valued *G*-Metric Spaces
C^*-*avmMS*	C^*-algebra valued modular Metric Spaces
C^*-*avmGMS*	C^*-algebra valued modular *G*-Metric Spaces

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